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Singularities of Jacobi Series on C^2 and the Poisson Process Equation

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A classical theorem of Gabor Szego relates the singularities of real zonal harmonic expansions with those of associated analytic functions of a single complex variable. Zeev Nehari developed the counterpart for Legendre series on the C-plane by generalizing Szego's theorem. This paper function theretically identifies the singularities of analytic symmetric Jacobi series on C^2 with those of analytic functions on the C-plane. One feature is that information about the singularities of solutions of Solomon Bochner's Poisson process equation flow from the expansion coefficients. Others are that the Szego and Nehari theorems appear on characteristic subspaces. And, that this PDE, unlike those normally encountered in function theory, is hyperbolic in the real domain. © 1987 Academic Press, Inc.

INTRODUCTION

A classical theorem of Nehari [8] develops the singularities of Legendre series on the C_z-plane from those of an associated function analytic on the C_t-plane. Szego's [9] antecedent characterizes the singularities of zonal harmonic expansions similarly. Their method is to apply the Hadamard argument in the multiplication of singularities theorem [6] to reciprocal integral equations connecting the series and the associate, and, thereby link the singularities of the two functions. Information about the location and structure of singularities of the more general functions then flows from the expansion coefficients as in the tradition of the theorems of Taylor, Mandelbrojt, and Fabry (see [3, 6]).

Function theory has broadened these ideas to the study of singularities of solutions to more general elliptic, as well as, parabolic partial differential equations. The envelope method of Gilbert (see [4–5]) is the focal point of the singularity theory. The envelope method builds on the reasoning in the Hadamard argument so that it applies to reciprocal transform pairs connecting solutions of these types of partial differential equations with associated analytic functions.

To take Nehari's theorem in other directions let us consider series expansions formed from terms that are products of Jacobi polynomials of degree n on the complex space $C^2 := C_z \times C_w$. These arise in the general theory of

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orthogonal functions on C^2. They form a basis for a subspace of symmetric analytic functions on C^2 and in turn for analytic solutions of the hyperbolic partial differential equation

$$[\partial_x[\rho_{\alpha,\beta}(x, y)\, \partial_x] - \partial_y[\rho_{\alpha,\beta}(y, x)\, \partial_y]]\, F(x, y) = 0$$
$$\rho_{\alpha,\beta}(x, y) := (1-x)^{\alpha}\,(1+x)^{\beta}\,(1-y)^{\alpha+1}\,(1+y)^{\beta+1}, \tag{1}$$
$$\alpha, \beta \geqslant -\tfrac{1}{2}, \alpha + \beta \geqslant -\tfrac{1}{2},$$

studied by Bochner [2] in the real domain as *Poisson processes* for $|x| \leqslant 1$ and $|y| \leqslant 1$.

The plan is to determine the singular manifolds of Poisson processes on C^2. To achieve this in a function theoretic context, the coordinates (x, y) are extended as independent complex variables (z, w) making the equation of mixed type on C^2. Then one constructs Cauchy type reciprocal integral transforms linking a Poisson process with its unique analytic associate in the C_t-plane. During the analytic continuation of the transform pair, the envelope method defines the sets of possible singularities of the function element and its associate. The actual singularities of the Poisson process, or those of its associate depending on one's viewpoint, are located precisely at the coincident set. As corollaries, the Nehari and Szego theorems and the Poisson processes on E^2, appear on characteristic subspaces of C^2. It is noted that even though the resulting transform pair $\{\mathbf{P}, \mathbf{P}^{-1}\}$ is between functions defined on the spaces $C_t \leftrightarrow C_z \times C_w$, the variables $(x, y) \leftrightarrow (z, w)$ are functionally related through the PDE so that the transform pair does not contain a singular map.

Preliminaries

A Poisson process analytic at the origin in E^2 admits a local expansion

$$F(x, y) = \sum_{n=0}^{\infty} \omega_n a_n P_n^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(y) \tag{2a}$$

in terms of the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ [10] which are orthogonal

$$\int_{-1}^{+1} P_n^{(\alpha,\beta)}(s)\, P_m^{(\alpha,\beta)}(s)\, d\mu_{\alpha,\beta}(s) = \omega_n^{-1}\, \delta_{nm},$$

$$\omega_n := (2n+\alpha+\beta+1)\, \Gamma(n+1)\, \Gamma(n+\alpha+\beta+1)/2^{\alpha+\beta+1}$$
$$\times \Gamma(n+\alpha+1)\, \Gamma(n+\beta+1) \sim n^{2\alpha+1},$$

relative to the measure

$$d\mu_{\alpha,\beta}(s) := \rho_{\alpha,\beta}(s, o)\, ds.$$

The first step is to associate F locally with the even analytic function

$$f(t) = \sum_{n=0}^{\infty} a_n t^{2n}, \qquad t \in C_t. \tag{2b}$$

Proceeding, let us define the formal *kernel*

$$K(s, z, w) := \sum_{n=0}^{\infty} s^{2n} P_n^{(\alpha,\beta)}(z)\, P_n^{(\alpha,\beta)}(w)$$

and the ellipse

$$E_\delta := \{\eta \in C: |\eta + 1| + |\eta - 1| < \delta\}$$

that has $2\delta > 0$ as the length of the major and minor axes.

On compacta of the polydomain $E_{\delta(z)} \times E_{\delta(w)} \subset C^2$, Darboux's formula [10] provides the estimate

$$\omega_n P_n^{(\alpha,\beta)}(z)\, P_n^{(\alpha,\beta)}(w) \sim n^{2\alpha+1}\, \delta^{2n},$$

where $\delta = \min\{\delta(z), \delta(w)\}$. Consequently, on sufficiently small compacta about the origin, the kernel is dominated by the derivative of a (absolutely and uniformly convergent) geometric series. This establishes its local existence as an analytic function on C^3 by Hartog's theorem [4].

Now reasoning along the lines of [7, 8], let the sequence $\{a_n\}_{n=0}^{\infty}$ be real and let

$$\limsup |a_n|^{1/n} := \sigma^{-1} \tag{3}$$

for $0 < \sigma < 1$. Given $\varepsilon > 0$, define the circular contour γ: $|t| = (\sigma + \varepsilon)^{-1}$ with positive orientation and apply Cauchy's theorem to show that

$$F(x, y) = (1/2\pi i) \int_\gamma f(t)\, K(t^{-1}, x, y)\, dt/t \tag{4a}$$

for $(x, y) \in E^2$ sufficiently near the origin. The inverse transformation

$$f(t) = \int_{-1}^{+1} \int_{-1}^{+1} F(x, y)\, K(t, x, y)\, d\mu_{\alpha,\beta}(x)\, d\mu_{\alpha,\beta}(y) \tag{4b}$$

follows from orthogonality of the Jacobi polynomials. The maximal domains of analyticity for the pair (4) are taken as the initial domains of definition.

To see that F admits a local analytic continuation to C^2, refer directly to the Darboux type estimate and Eq. (3). The analytic continuation of $F(x, y)$ to $F(z, w)$ extends the pair (4) onto the domains of association (see [4–5]), where the transforms

$$F(z, w) = (1/2\pi i) \int_{L_t} f(t)\, K(t^{-1}, z, w)\, dt/t \tag{5a}$$

$$f(t) = \int_{L_z} \int_{L_w} F(z, w)\, K(t, z, w)\, d\mu_{\alpha,\beta}(z)\, d\mu_{\alpha,\beta}(w) \tag{5b}$$

link them as the principal value function elements. The contours are simple; L_t is closed and homologous to the contour γ, whereas L_z and L_w are open contours from -1 to $+1$ that are homologous to the segment $[-1, +1]$. Each homology is modulo the singularities of the respective integrand.

The Reciprocal Integral Equations

A useful formulation of (5) requires a closed representation of the kernel. This is available by noting that Watson [11, pp. 27–28] represents the kernel as a generating function. We rewrite this formula as the contour integral

$$K(t, z, w) = \int_{L_\tau} W(a, b, t, \tau)\, d\tau/\tau,$$

$$2a := (1-z)^{1/2}\,(1-w)^{1/2}, \qquad 2b := (1+z)^{1/2}\,(1+w)^{1/2},$$

where the simple contour L_τ joins 0 to 1 and is homologous to the segment $[0, 1]$.

Working toward our definition of the Watson kernel, W, let a and b be real. And let the vector $\xi = \xi(a, b, k)$ be the analytic continuation of the root of least modulus and smallest nonnegative argument of the biquadratic equation

$$\xi\colon (k^2 + a^2 z^2)(k^2 + b^2 z^2) - k^4 z^2 = 0, \qquad 2k = t + t^{-1}.$$

The continuation corresponds to the initial point $(x, y) = (0, 0)$ taken as (x, y) moves into (z, w) over the domain of association and as the segment $[0, 1]$ is deformed to L_τ.

The generating variables are defined [11, p. 28] as

$$\sigma(\mu, v) := \mu^2 + v^2, \qquad \Omega^2(\mu, v, \lambda, \eta) = \sigma(\mu, v)\,\sigma(\mu, \lambda) - \eta^2,$$
$$\cos^2(\omega) := \eta^2/(\eta^2 + \Omega^2).$$

The *Watson kernel* [11, p. 28] is rewritten as

$$W(a, b, t, \tau) = c(t, \tau)\,\partial_k[F_1 F_2 F_3]\,|_{(a,b,t,\tau)}$$
$$c(t, \tau) := -(1 - t^2)/\pi(2t)^{\alpha+\beta+1},$$

where $[F_1 F_2 F_3]$ represents the product of the functions

$$F_1(\mu, v, \lambda, v) := \mu(\xi\tau/\mu)^{\alpha+\beta}\,(\sigma(\mu, \lambda)/\sigma(\mu, v))^{(\alpha+\beta)/2}$$
$$F_2(\mu, v, \lambda, v) := \cos((\alpha - \beta)\omega), \qquad F_3(\mu, v, \lambda, v) := \Omega^{-1}(\mu, v, \lambda, \eta)$$

whose arguments are

$$(\mu, v, \lambda, \eta) := (k, a\xi\tau, k^2\tau\xi)|_{2k = t + t^{-1}}.$$

An explicit expression for the cosine term is not require for the analysis of the singularities. The symmetry relations

$$W(a, b, t^{-1}, \tau) = -t^{2\alpha+2\beta-1}W(a, b, t, \tau)$$
$$K(t^{-1}, z, w) = -t^{2\alpha+2\beta-1}K(t, z, w)$$

are useful in summarizing the following regularity theorem.

THEOREM 1. *The reciprocal integral equations*

$$F(z, w) = \mathbf{P}(f(t))$$
$$:= (-1/2\pi i)\int_{L_\tau}\int_{L_t} f(t)\,t^{2\alpha+2\beta-1}W(a, b, t, \tau)\,dt\,d\tau/\tau \tag{6a}$$
$$f(t) = \mathbf{P}^{-1}(F(z, w))$$
$$:= \int_{L_\tau}\int_{L_w}\int_{L_z} F(z, w)\,W(a, b, t, \tau)(1 - z)^\alpha$$
$$\times (1 + z)^\beta (1 - w)^\alpha (1 + w)^\beta\,dz\,dw\,d\tau/\tau, \tag{6b}$$

where $\alpha, \beta \geqslant -\frac{1}{2}$, $\alpha + \beta \geqslant -\frac{1}{2}$, *form a dual transform pair that uniquely link the function elements F and f on their domains of association.*

A transform pair analogous to (6) maybe developed using the generating function found in Bailey [1, p. 11]. Both the Watson and Bailey formulations of the generating function produce transforms whose kernels are complicated. Nevertheless, the generating variables in (6) enter in just the right way (they do not appear as arguments of the associates) and calculation of the manifolds of possible singularities is relatively straightforward.

THE SINGULAR MANIFOLDS

The analytic continuation of the function elements generates motion of the singularities of the integrands of the transform pair. During the process the domain of association is constructed: in so doing, the contours of integration are deformed to avoid singularities that approach them along an intersecting trajectory. The envelope method describes in a precise analytical way those points for which no deformation will avoid an intersection. Such a point is located on the set of possible singularities of the function element; or it is an "end pinch" singularity that arises because the terminal points of an open contour are fixed and are not permitted motion to escape an intersection.

Let us begin with a Poisson process F whose associate has a singularity at $t = t_0 \neq 0$. We may write the analytic manifold of possible singularities as

$$M(z, w, t, \tau) = M(z_0, w_0, t_0, \tau_0) + \nabla M(z_0, w_0, t_0, \tau_0) \cdot (z - z_0, w - w_0, t - t_0, \tau - \tau_0) + \cdots, \tag{7}$$

where $(z_0, w_0, -, \tau_0)$ is in correspondence with a possible intersection. Upon expanding the kernel and examining each of the terms, the envelope method creates the manifolds (and derived manifolds) of possible singularities of the transform (6a) as

$$G = (G_1 \cap G_1^*) \cup (G_2 \cap G_2^*) \cup (G_3 \cap G_3^*)$$

where

$$G_1(f; t_0) = \{(z, w) \in C^2 \colon \Omega^2(k, a\xi\tau, b\xi\tau, k^2\xi\tau) = 0; \tau \in L_\tau, t = t_0\}$$

$$G_1^*(f; t_0) = \{(z, w) \in C^2 \colon \partial_\tau(\Omega^2(k, a\xi\tau, b\xi\tau, k^2\xi\tau) + k^2\xi\tau) = 0; \tau \in L_\tau, t = t_0\}$$

$$G_2(f; t_0) = \{(z, w) \in C^2 \colon \Omega^2(k, a\xi\tau, b\xi\tau, k^2\xi\tau) + k^4\xi^2\tau^2 = 0; \tau \in L_\tau, t = t_0\}$$

$$G_2^*(f; t_0) = \{(z, w) \in C^2 \colon \partial_\tau(\Omega^2(k, a\xi\tau, b\xi\tau, k^4\xi^2\tau^2)) = 0; \tau \in L_\tau, t = t_0\}$$

$$G_3(f; t_0) = \{(z, w) \in C^2 \colon k^2 + a^2\xi^2\tau^2 = 0; \tau \in L_\tau, t = t_0\}$$

$$G_3^*(f; t_0) = \{(z, w) \in C^2 \colon \partial_\tau(k^2 + a^2\xi^2\tau^2) = 0; \tau \in L_\tau, t = t_0\}.$$

A few remarks are in order. The manifold G_3 arises by eliminating the variable ω from the factor F_2 in the kernel and observing that the result is an entire function of its argument. In other words the factor generates the possible singularities only when the argument is unbounded. Moreover, it is assumed that the parameters $\alpha \neq \beta$, otherwise $F_3 \equiv 1$ and $G_2 = G_2^* = \varnothing$. Excluding the endpoint singularities for the moment, we find that

$$G_1: (k^2 + a^2\xi^2\tau^2)(k^2 + b^2\xi^2\tau^2) - k^4\xi^2\tau^2 = 0$$

$$G_1^*: a^2(k^2 + b^2\xi^2\tau^2) + b^2(k^2 + a^2\xi^2\tau^2) - k^4 = 0$$

$$G_2: (k^2 + a^2\xi^2\tau^2)(k^2 + b^2\xi^2\tau^2) = 0$$

$$G_2^*: a^2(k^2 + b^2\xi^2\tau^2) + b^2(k^2 + a^2\xi^2\tau^2) = 0$$

$$G_3: k^4 + a^2\xi^2\tau^2 = 0, \qquad G_3^*: 2a^2\xi^2\tau = 0.$$

Eliminating the parameter τ from these sets gives

$$\begin{aligned} G_1(f; t_0) \cap G_1^*(f; t_0) = \{(z, w) \in C^2 : \sigma^2 - (1 + zw)^2 \\ = (1 - z^2)(1 - w^2),\ t^2 - 2\sigma t + 1 = 0\} \end{aligned}$$

$$G_2(f; t_0) \cap G_2^*(f; t_0) = \varnothing$$

$$G_3(f; t_0) \cap G_3^*(f; t_0) = \{(z, w) \in C^2 : z = \pm w\}.$$

The manifold $G_3 \cap G_3^*$ passes thru the origin, a regular point of F, and it cannot represent a singularity of the principal branch. This leaves the manifold

$$G(f; t_0) = G_1(f; t_0) \cap G_1^*(f; t_0)$$

as the only possible singular set of $F = \mathbf{P}f$. The Poisson process F is analytic on the complement. Note that Hadamard singularities (see [4–5]) are located on G a priori.

Let us turn the argument around and examine the manifold of possible singularities of the inverse transform $f = \mathbf{P}^{-1}F$. Assume that F has a singularity at $(z_0, w_0) \neq (0, 0)$ and expand the singular manifold $M(z, w, t, \tau)$ as in Eq. (7). The arguments of the associates do *not* contain generating variables and the kernel of each transformation is essentially identical (except for the weights). Applying the envelope method to the point $(-, -, t_0, \tau_0)$ in correspondence to a possible singularity produces the manifold and derived manifold of the inverse as

$$M = (M_1 \cap M_1^*) \cup (M_2 \cap M_2^*) \cup (M_3 \cap M_3^*),$$

where

$$M_1(F; (z_0, w_0)) = \{t \in C_t: \Omega^2(k, a\xi\tau, b\xi\tau, k^2\xi\tau) = 0;$$
$$\tau \in L_\tau, (z, w) = (z_0, w_0)\}$$
$$M_1^*(F; (z_0, w_0)) = \{t \in C_t; \partial_\tau \Omega^2(k, a\xi\tau, b\xi\tau, k^2\xi\tau) = 0;$$
$$\tau \in L_\tau, (z, w) = (z_0, w_0)\}$$
$$M_2(F; (z_0, w_0)) = \{t \in C_t: \Omega^2(k, a\xi\tau, b\xi\tau, k^2\xi\tau) + k^4\xi^2\tau^2 = 0;$$
$$\tau \in L_\tau, (z, w) = (z_0, w_0)\}$$
$$M_2^*(F; (z_0, w_0)) = \{t \in C_t: \partial_\tau(\Omega^2(k, a\xi\tau, b\xi\tau, k^2\xi\tau) + k^4\xi^2\tau^2) = 0;$$
$$\tau \in L_\tau, (z, w) = (z_0, w_0)\}$$
$$M_3(F; (z_0, w_0)) = \{t \in C_t: k^2 + a^2\xi^2\tau^2 = 0; \tau \in L_\tau, (z, w) = (z_0, w_0)\}$$
$$M_3^*(F; (z_0, w_0)) = \{t \in C_t: \partial_\tau(k^2 + a^2\xi^2\tau^2) = 0; \tau \in L_\tau, (z, w) = (z_0, w_0)\}.$$

Proceeding along the same lines we find that

$$M_1(F; (z_0, w_0)) \cap M_1^*(F; (z_0, w_0))$$
$$= \{t \in C_t: \sigma^2 - (1 + z_0 w_0)^2 = (1 - z_0^2)(1 - w_0^2), t^2 - 2\sigma t + 1 = 0\}$$
$$M_2(F; (z_0, w_0)) \cap M_2^*(F; (z_0, w_0)) = \varnothing,$$

and

$$M_3(F; (z_0, w_0)) \cap M_3^*(F; (z_0, w_0)),$$

where the intersection of M_3 and M_3^* is either empty or it is a subset of the intersection of M_1 and M_1^*.

Checking the endpinch singularities of each transform at the branch manifolds

$$E_0 = \{(z, w) \in C^2: 1 - z^2 = 0\} \cup \{(z, w) \in C^2: 1 - w^2 = 0\}$$

and

$$E_1 = \{\tau \in C: \tau = 0, 1\}$$

on the manifold $M_1 \cap M_1^*$ produces either the Nehari points

$$\{(\pm 1, 1 \pm (t_0 + t_0^{-1})/2)\} \cup \{(1 \pm (t_0 + t_0^{-1})/2, \pm 1)\}$$

or $M_1 \cap M_1^* = \varnothing$. We have completed the analysis.

THEOREM 2. *Let the real analytic expansions*

$$F(z, w) = \sum_{n=0}^{\infty} \omega_n a_n P_n^{(\alpha,\beta)}(z) P_n^{(\alpha,\beta)}(w), \qquad \alpha, \beta \geqslant -\tfrac{1}{2}, \alpha + \beta \geqslant -\tfrac{1}{2}$$

$$f(t) = \sum_{n=0}^{\infty} a_n t^{2n}$$

define functions on their initial domains of definition. Then the point $(z_0, w_0) \in C^2$ *reached by analytically continuing the function element F and avoiding the Nehari points*

$$(\pm 1, 1 \pm (t_0 + t_0^{-1})/2) \qquad \text{and} \qquad (1 \pm (t_0 + t_0^{-1})/2, \pm 1)$$

is a singularity if, and only if,

$$\sigma^2 - (1 + z_0 w_0)^2 = (1 - z_0^2)(1 - w_0^2), \sigma = (t_0 + t_0^{-1})/2,$$

where $t = t_0 \in C_t$ *is a singularity reached by analytically continuing the function element f.*

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